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LETTER TO THE EDITOR

On the Bäcklund transformation for the Moyal Korteweg–de Vries hierarchy

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Abstract
We study the Bäcklund symmetry for the Moyal Korteweg–de Vries (KdV) hierarchy based on the Kupershmidt–Wilson theorem associated with second Gelfand–Dickey structure with respect to the Moyal bracket, which generalizes the result of Adler for the ordinary KdV.

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In [1], Adler provided an elegant approach to the Bäcklund transformation for the generalized Korteweg–de Vries (KdV) equations [2, 3] by exploring the hidden symmetry of the associated modified KdV equations. The approach is intimately related to the Kupershnidt–Wilson (KW) theorem [4] (see also [5, 6]) for the second Gelfand–Dickey (GD) bracket [2, 3], the Hamiltonian structure of the KdV, in which the Miura variables play an important role, based upon which some generalizations including Toda [1], Drinfeld–Sokolov [7], Kadomtsev–Petviashvili [8–10], supersymmetric KdV [11] and q-deformed KdV [12] hierarchies can be formulated in a similar way.

In this Letter, we should like to generalize the approach to the second GD structure with respect to the Moyal bracket [13] defined by

$$\{f, g\}_\kappa = \frac{f \star g - g \star f}{2\kappa} \tag{1}$$

where f and g are two arbitrary functions on the two-dimensional phase space with coordinate (x, p) and the $\star$-product [14] is defined by

$$f \star g = \sum_{s=0}^{\infty} \frac{\kappa^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} \left(\partial_x^j \partial_p^{s-j} f\right)\left(\partial_x^{s-j} \partial_p^j g\right). \tag{2}$$

The Moyal bracket (1) can be viewed as the higher-order derivative generalization of the canonical Poisson bracket since it recovers the canonical Poisson bracket in the dispersionless

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limit $\kappa \to 0$, i.e. $\lim_{\kappa\to 0}\{f, g\}_\kappa = \partial_p f \partial_x g - \partial_x f \partial_p g$. Therefore the Lax equations defined by the Moyal bracket can be viewed as a one-parameter deformation of dispersionless Lax equations (see [15] for a review) defined by the canonical Poisson bracket.

Our main results contain two parts. In theorem A, we prove the KW theorem for the second GD structure with respect to the Moyal bracket, which simplifies the associated Hamiltonian flows in terms of the Miura variables. We then, in theorem B, show that the Bäcklund transformation for the generalized Moyal KdV hierarchy can be traced back to the permutation symmetry of the Miura variables via the KW theorem and thus generalize Adler's work to the present case.

To begin with, we consider an algebra of Laurent series of the form $\Lambda = \{A|A = \sum_{i=-\infty}^{N} a_i p^i\}$ with coefficients a_i depending on an infinite set of variables $t_1 \equiv x, t_2, t_3, \ldots$. The algebra Λ with respect to the Moyal bracket (1) can be decomposed into the subalgebras as $\Lambda_+ \oplus \Lambda_-$, where the subscript + stands for the projection onto the non-negative powers in p. It is obvious that Λ is an associative but noncommutative algebra under the $\star$-product. For a given Laurent series $A = \sum_i a_i p^i$ one defines its residue as $\mathrm{res}(A) = a_{-1}$ and its trace as $\mathrm{tr}(A) = \int \mathrm{res}(A)$. For any two Laurent series $A = \sum_i a_i p^i$ and $B = \sum_j b_j p^{-j}$ it is easy to show that (i) $\int \mathrm{res}(A \star B) = \sum_i \int a_i b_{i+1}$, (ii) $\mathrm{tr}\{A, B\}_\kappa = 0$ and (iii) $\mathrm{tr}(A \star \{B, C\}_\kappa) = \mathrm{tr}(\{A, B\}_\kappa \star C)$. Given a functional $F(A) = \int f(a)$ we define its gradient as $\delta F/\delta A \equiv d_A F = \sum_i \delta F/\delta a_i\, p^{-i-1}$, where the variational derivative is defined by $\delta F/\delta a_k = \sum_i (-\partial_x)^i (\partial f/\partial a_k^{(i)})$, with $a_k^{(i)} \equiv \partial_x^i a_k$, $\partial \equiv \partial/\partial x$. Note that we use the notations $\partial_x f = f' = f_x$ throughout this paper.

Let us consider the Lax equations

$$\frac{\partial L}{\partial t_k} = \{(L^{1/n}\star)^k_+, L\}_\kappa \qquad (L^{1/n}\star)^k_+ = (\underbrace{L^{1/n} \star L^{1/n} \star \cdots \star L^{1/n}}_{k})_+ \tag{3}$$

where the Lax operator $L = p^n + \sum_{i=0}^{n-2} u_i p^i$ is a polynomial in p and $L^{1/n} = p + \sum_{i=1} a_i p^{-i}$ is the nth root of L in such a way that $L = (L^{1/n}\star)^n$. The Lax equation (3) defines the dynamical flows for u_i, which form what we call the generalized Moyal KdV hierarchy. Note that the highest order in p on the right-hand side of the Lax equations (3) is $n-2$, and thus one can drop the term u_{n-1} in the Lax formulation without causing any problem. However, we shall see that imposing the constraint $u_{n-1} = 0$ induces a modification in the Hamiltonian formulation.

For the polynomial $L = p^n + \sum_{i=0}^{n-1} u_i p^i$ and functionals $F[L]$ and $G[L]$ we define the second GD bracket [2, 3] with respect to the $\star$-product as

$$\{F, G\}_2(L) = \mathrm{tr}\,[J_2(d_L F) \star d_L G] \tag{4}$$

in which J_2 is the Adler map [16] defined by

$$J_2(X) = \{L, X\}_{\kappa +} \star L - \{L, (X \star L)_+\}_\kappa \tag{5}$$

where $X = \sum_{i=1}^{n} x_i p^{-i-1}$. The bracket (4) is anti-symmetric due to the cyclic property of the trace and satisfies the Jacobi identity that will be justified later on. In the $\kappa \to 0$ limit (4) recovers the dispersionless GD bracket [17].

Theorem A (Kupershmidt and Wilson). *Let* $L = p^n + \sum_{i=0}^{n-1} u_i p^i = A \star B$ *where* $A = p^m + \sum_{i=0}^{m-1} a_i p^i$ *and* $B = p^l + \sum_{i=0}^{l-1} b_i p^i$ *and* $m + l = n$*; then we have*

$$\{F, G\}_2(L) = \{F, G\}_2(A) + \{F, G\}_2(B). \tag{6}$$

Proof. From the variation

$$\begin{aligned}\delta F &= \mathrm{tr}(d_L F \star \delta L)\\ &= \mathrm{tr}(d_L F \star \delta A \star B + d_L F \star A \star \delta B)\\ &= \mathrm{tr}(d_A F \star \delta A) + \mathrm{tr}(d_B F \star \delta B)\end{aligned}$$

we have $d_A F = B \star d_L F$ and $d_B F = d_L F \star A$. Then

$$\begin{aligned}\mathrm{RHS} &= \mathrm{tr}[\{A, d_A F\}_{\kappa+} \star A \star d_A G - \{A, (d_A F \star A)_+\}_\kappa \star d_A G] + (A \leftrightarrow B)\\ &= \frac{1}{2\kappa}\mathrm{tr}[(A \star d_A F)_+ \star A \star d_A G - (d_A F \star A)_+ \star A \star d_A G)\\ &\quad - A \star (d_A F \star A)_+ \star d_A G + (d_A F \star A)_+ \star A \star d_A G] + (A \leftrightarrow B)\\ &= \frac{1}{2\kappa}\mathrm{tr}[(A \star d_A F)_+ \star A \star d_A G - A \star (d_A F \star A)_+ \star d_A G] + (A \leftrightarrow B)\\ &= \mathrm{tr}[J_2(d_L F) \star d_L G].\end{aligned}$$

□

If we factorize the Lax operator as

$$L = p^n + \sum_{i=0}^{n-1} u_i p^i = (p - \phi_n) \star \cdots \star (p - \phi_1) \tag{7}$$

then the coefficients u_i can be expressed in terms of Miura variables ϕ_i as

$$\begin{aligned}u_{n-1} &= -\sum_{i=1}^{n} \phi_i\\ u_{n-2} &= \sum_{i>j} \phi_i \phi_j - \kappa \sum_{i=1}^{n} (n - 2\mathrm{i} + 1)\phi_i'\\ &\vdots\end{aligned} \tag{8}$$

which constitute the Miura transformation. Therefore under the factorization (7), by theorem A, the second GD structure (4) can be simplified as follows:

$$\{F, G\}_2(L) = \sum_{i=1}^{n} \int \left(\frac{\delta F}{\delta \phi_i}\right)' \left(\frac{\delta G}{\delta \phi_i}\right).$$

In particular,

$$\{\phi_i(x), \phi_j(y)\}_2 = -\delta_{ij}\partial_x \delta(x - y) \tag{9}$$

which immediately verifies the Jacobi identity of the GD bracket (4).

The second Hamiltonian structure for the generalized Moyal KdV hierarchy can be obtained from (4) by imposing the constraint $u_{n-1} = -\sum_{i=1}^{n} \phi_i = 0$. Using (9) we obtain $\{u_{n-1}(x), u_{n-1}(y)\}_2 = -n\partial_x \delta(x - y)$, which has an inverse for $n \neq 0$ and hence the constraint is second class. The standard Dirac procedure thus gives the modified Adler map as

$$J_2^D(X) = \{L, X\}_{\kappa+} \star L - \{L, (X \star L)_+\}_\kappa + \frac{1}{n}\left\{L, \int^x \mathrm{res}\{L, X\}_\kappa\right\}_\kappa \tag{10}$$

or, in terms of Miura variables,

$$\{\phi_i(x), \phi_j(y)\}_2^D = \left[\frac{1}{n} - \delta_{ij}\right]\partial_x \delta(x - y). \tag{11}$$

On the other hand, one might obtain the associated first GD structure for the Moyal KdV by shifting $L \to L + \lambda$ in (10) and then extract the term linear in λ [3]. It turns out that

$$J_1(X) = \{L, X\}_{\kappa+} \tag{12}$$

which is compatible with the reduction $u_{n-1} = 0$. As a result, the bi-Hamiltonian flows for the generalized Moyal KdV hierarchy can be written as

$$\frac{\partial L}{\partial t_k} = J_2^D(d_L H_k) = J_1(d_L H_{k+n}) \tag{13}$$

with

$$H_k = \frac{n}{k}\int \mathrm{res}(L^{1/n}\star)^k.$$

Next we turn to the Bac̆klund transformation for the Moyal KdV hierarchy. Our strategy is to rewrite the hierarchy flows in terms of the Miura variables. Following Adler [1] we define the permutation $\Omega : \phi_1 \to \phi_2, \phi_2 \to \phi_3, \ldots, \phi_n \to \phi_1$, then $L_{\Omega^i} = (p - \phi_i) \star (p - \phi_{i-1}) \star \cdots \star (p - \phi_{i+1})$. In particular, $L_{\Omega^n} = L_{\Omega^0} = L$ and $L_{\Omega^i} = p_i \star L_{\Omega^{i-1}} \star p_i^{-1}$ where $p_i \equiv p - \phi_i$ and its inverse p_i^{-1} can be expressed as $\exp(\int^x \phi_i/2\kappa) \star p \star \exp(-\int^x \phi_i/2\kappa)$ and $\exp(\int^x \phi_i/2\kappa) \star p^{-1} \star \exp(-\int^x \phi_i/2\kappa)$, respectively.

Theorem B (Adler). *The Miura variables ϕ_i satisfy the following modified Moyal KdV equations:*

$$\frac{\partial \phi_i}{\partial t_k} = \frac{1}{2\kappa}\left[p_i \star B^{(k)}_{\Omega^{i-1}} - B^{(k)}_{\Omega^i} \star p_i\right] \tag{14}$$

where $B^{(k)}_{\Omega^i} \equiv (L^{1/n}_{\Omega^i}\star)^k_+$.

Before proving (14) we observe that (14) is compatible with the Lax flows (3) since from the factorization form of L we have

$$\begin{aligned}\frac{\partial L}{\partial t_k} &= -\sum_{i=1}^{n} p_n \star \cdots \star p_{i+1} \star \frac{\partial \phi_i}{\partial t_k} \star p_{i-1} \star \cdots \star p_1 \\ &= -\frac{1}{2\kappa}\sum_{i=1}^{n} p_n \star \cdots \star p_{i+1} \star \left(p_i \star B^{(k)}_{\Omega^{i-1}} - B^{(k)}_{\Omega^i} \star p_i\right) \star p_{i-1} \star \cdots \star p_1 \\ &= \{B^{(k)}, L\}_\kappa.\end{aligned}$$

Lemma. *The Hamiltonian flows for the Miura variables ϕ_i can be expressed as*

$$\frac{\partial \phi_i}{\partial t_k} = \mathrm{res}\left\{p_i, p_{i-1} \star \cdots \star p_1 \star (L^{1/n}\star)^{k-n} \star p_n \star \cdots \star p_{i+1}\right\}_\kappa. \tag{15}$$

Proof. From the variation

$$\begin{aligned}\frac{\delta L}{\delta \phi_i} = &\frac{\delta L^{1/n}}{\delta \phi_i} \star L^{1/n} \star \cdots \star L^{1/n} \\ &+ L^{1/n} \star \frac{\delta L^{1/n}}{\delta \phi_i} \star \cdots \star L^{1/n} \\ &\vdots \\ &+ L^{1/n} \star L^{1/n} \star \cdots \star \frac{\delta L^{1/n}}{\delta \phi_i}\end{aligned}$$

we have

$$\begin{aligned}\frac{\delta H_k}{\delta \phi_i} &= n\,\mathrm{tr}\left[\frac{\delta L^{1/n}}{\delta \phi_i} \star (L^{1/n}\star)^{k-1}\right] \\ &= \mathrm{tr}\left[\frac{\delta L}{\delta \phi_i} \star (L^{1/n}\star)^{k-n}\right] \\ &= -\mathrm{res}\left[p_{i-1} \star \cdots \star p_1 \star (L^{1/n}\star)^{k-n} \star p_n \star \cdots \star p_{i+1}\right].\end{aligned}$$

Hence

$$\frac{\partial\phi_i}{\partial t_k} = \{\phi_i, H_k\}_2^D = \sum_j \left[\frac{1}{n} - \delta_{ij}\right]\left(\frac{\delta H_k}{\delta\phi_j}\right)'$$
$$= -\sum_j \left[\frac{1}{n} - \delta_{ij}\right]\big\{p_j, \mathrm{res}[p_{j-1} \star \cdots \star p_1 \star (L^{1/n}\star)^{k-n} \star p_n \star \cdots \star p_{j+1}]\big\}_\kappa$$
$$= \mathrm{res}\big\{p_i, p_{i-1} \star \cdots \star p_1 \star (L^{1/n}\star)^{k-n} \star p_n \star \cdots \star p_{i+1}\big\}_\kappa.$$

□

Proof of theorem B. From the relation $L_{\Omega^i} = p_i \star L_{\Omega^{i-1}} \star p_i^{-1}$ we have

$$\begin{aligned} B_{\Omega^i}^{(k)} &= \big(L_{\Omega^i}^{1/n}\star\big)_+^k \\ &= \big(p_i \star B_{\Omega^{i-1}}^{(k)} \star p_i^{-1}\big)_+ \\ &= p_i \star B_{\Omega^{i-1}}^{(k)} \star p_i^{-1} - \big(p_i \star B_{\Omega^{i-1}}^{(k)} \star p_i^{-1}\big)_-. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2\kappa}\big[p_i \star B_{\Omega^{i-1}}^{(k)} - B_{\Omega^i}^{(k)} \star p_i\big] &= \frac{1}{2\kappa}\big(p_i \star B_{\Omega^{i-1}}^{(k)} \star p_i^{-1}\big)_- \star p_i \\ &= \frac{1}{2\kappa}\mathrm{res}\big(p_i \star B_{\Omega^{i-1}}^{(k)} \star p_i^{-1}\big) \\ &= \frac{1}{2\kappa}\big[\mathrm{res}\big(p_i \star (L_{\Omega^{i-1}}^{1/n}\star)^k \star p_i^{-1}\big) - \mathrm{res}\big(L_{\Omega^{i-1}}^{1/n}\star\big)^k\big] \\ &= \mathrm{res}\big\{p_i, p_{i-1} \star \cdots \star p_1 \star (L^{1/n}\star)^{k-n} \star p_n \star \cdots \star p_{i+1}\big\}_\kappa \end{aligned}$$

which, by lemma, completes the proof of theorem B. □

We emphasize here that (15) (and hence (14)) is consistent with the constraint $u_{n-1} = -\sum_i \phi_i = 0$.

Thus the cyclic permutation Ω generates the Bäcklund transformation of the hierarchy due to the fact that the form of the Lax operator and the hierarchy flows are preserved under such transformation. In particular, the one-step permutation Ω: $L_{\Omega^{i-1}} \to L_{\Omega^i}$ defines an elementary Bäcklund transformation. Indeed, if $L_{\Omega^{i-1}}$ satisfies (3), then the transformed Lax operator L_{Ω^i} satisfies

$$\begin{aligned} \frac{\partial L_{\Omega^i}}{\partial t_k} &= \left\{p_i \star B_{\Omega^{i-1}}^{(k)} \star p_i^{-1} + 2\kappa\frac{\partial p_i}{\partial t_k} \star p_i^{-1}, L_{\Omega^i}\right\}_\kappa \\ &= \left\{p_i \star B_{\Omega^{i-1}}^{(k)} \star p_i^{-1} - 2\kappa\frac{\partial \phi_i}{\partial t_k} \star p_i^{-1}, L_{\Omega^i}\right\}_\kappa \\ &= \big\{B_{\Omega^i}^{(k)}, L_{\Omega^i}\big\}_\kappa \end{aligned}$$

where (14) was used to reach the last line.

Finally let us work out the simplest example to illustrate the obtained results. For $n = 2$ we have $L = p^2 + u$ and the first few Lax flows are given by

$$\begin{aligned} u_{t_1} &= u_x \\ u_{t_3} &= \tfrac{3}{2}uu_x + \kappa^2 u_{xxx} \\ u_{t_5} &= \tfrac{15}{8}u^2u_x + \tfrac{5}{2}\kappa^2(uu_{xxx} + 2u_xu_{xx}) + \kappa^4 u^{(5)} \cdot \\ &\vdots \end{aligned} \tag{16}$$

The set of equations (16) is referred to as the Moyal KdV hierarchy [18–22]. For the Hamiltonian formulation, we can read off the Poisson brackets from (13) as [21, 22]

$$\begin{aligned} \{u(x), u(y)\}_1 &= 2\partial_x \delta(x-y) \\ \{u(x), u(y)\}_2^D &= [2\kappa^2\partial_x^3 + 2u\partial_x + u_x]\delta(x-y). \end{aligned} \tag{17}$$

The first few Hamiltonians for the Moyal KdV are

$$H_1 = \int u \qquad H_3 = \tfrac{1}{4}\int u^2 \qquad H_5 = \tfrac{1}{8}\int (u^3 + 2\kappa^2 u u_{xx})$$

$$H_7 = \tfrac{1}{64}\int (5u^4 - 40\kappa^2 u u_x^2 + 16\kappa^4 u_{xx}^2)$$

which together with (17) provides the bi-Hamiltonian flows

$$\frac{\partial u}{\partial t_{2n+1}} = [2\kappa^2\partial_x^3 + 2u\partial_x + u_x]\frac{\delta H_{2n+1}}{\delta u} = 2\partial_x \frac{\delta H_{2n+3}}{\delta u}.$$

We remark that when $\kappa = 0$ the Moyal KdV hierarchy reduces to the dispersionless KdV hierarchy [23] since all higher-order derivative terms disappear. On the other hand, (16) collapses to the ordinary KdV for $\kappa = 1/2$ due to an isomorphism between them [24] and thus the Moyal parameter κ characterizes a kind of dispersion effect.

Furthermore consider the factorization of the Lax operator $L = p^2 + u = (p-\phi)\star(p+\phi)$, which gives the Miura transformation (or Riccati relation) $u(\phi) = -\phi^2 + 2\kappa\phi'$. Then the Poisson algebra (17) can be easily rederived by using that bracket of the Miura variable (free-field) ϕ. Permuting the Miura variable ($\phi \to -\phi$) gives a new Lax operator $L_\Omega = p^2 + u_\Omega = (p+\phi)\star(p-\phi)$ with $u_\Omega(\phi) = u(\phi) - 4\kappa\phi'$. Now suppose the solutions of the Moyal KdV equations can be parametrized by a single function τ, the so-called tau-function, such that $u(x;t) = 8\kappa^2\partial_x^2 \ln\tau(x;t)$. Then the Adler approach to the Bäcklund transformation leads to the following transformation rule for tau functions:

$$\tau(x;t) \to \tau_\Omega(x;t) = \exp\left[-\frac{1}{2\kappa}\int^x \phi\right]\cdot\tau(x;t) \tag{18}$$

which reduces to the ordinary case [25] when $\kappa = 1/2$.

In summary, we have investigated the Bäcklund transformation for the Moyal KdV hierarchy from the viewpoint of the KW theorem. It turns out that the cyclic permutations of the Miura variables provide the elementary Bäcklund transformations of the hierarchy which can be expressed in terms of tau functions. It would be interesting to see whether the Moyal KdV hierarchy (16) has corresponding Hirota bilinear equations with (18) as a symmetry.

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